

q -Deformation of the Krichever-Novikov Algebra

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Abstract

Using q -operator product expansions between $U(1)$ current fields and also the corresponding energy-momentum tensors, we furnish the q -analogues of the generalized Heisenberg algebra and the Krichever-Novikov algebra.

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The recent focus on deformations of algebras called quantum algebras can be attributed to the fact that they appear to be the basic algebraic structures underlying an amazingly diverse set of physical situations. To date many interesting features of these algebras have been found and they are now known to belong to a class of algebras called Hopf algebras [1]. The remarkable aspect of these structures is that they can be regarded as deformations of the usual Lie algebras. Of late, there has been a considerable interest in the deformation of the Virasoro algebra and the underlying Heisenberg algebra [2-11]. In this letter we focus our attention on deforming generalizations of these algebras, namely the Krichever-Novikov (KN) algebra and its associated Heisenberg algebra.

The introduction by Krichever and Novikov [12-14] of a basis for the general tensor field over Riemann surfaces of arbitrary genus has led to a new operator formalism which generalizes conformal field theory on these surfaces. Indeed, on a surface Σ_g of genus g , the relevant algebra to consider is the centrally extended KN algebra:

$$[\mathcal{L}_m, \mathcal{L}_n] = \sum_{s=m+n-g_0}^{m+n+g_0} c_{m,n}^s \mathcal{L}_{m+n-s} + c\chi_{mn} \quad (1)$$

where $g_0 = 3g/2$. Here the indices m, n take integral values if g is even and half integral values for odd g . The structure constants $c_{m,n}^s$ and the cocycle χ_{mn} are evaluated using the KN bases for the vector fields $\{e_n\}$ and the quadratic differentials $\{\Omega_n\}$. Specifically one can introduce local coordinates z_+ and z_- around two distinguished points P_+ and P_- with the basis elements given by

$$e_n \equiv e_n(z_{\pm}) \frac{\partial}{\partial z_{\pm}} = \sum_{m=0}^{\infty} e_{n,m}^{\pm} z_{\pm}^{\pm n - g_0 + 1 + m} \frac{\partial}{\partial z_{\pm}} \quad (2a)$$

$$\Omega_n \equiv \Omega_n(z_{\pm}) dz_{\pm}^2 = \sum_{m=0}^{\infty} \Omega_{n,m}^{\pm} z_{\pm}^{\mp n + g_0 - 2 + m} (dz_{\pm})^2 \quad (2b)$$

in the neighbourhood of P_{\pm} . These two bases are dual in the sense

$$\pm \oint_{C_{\pm}} \frac{dz_{\pm}}{2\pi i} e_m(z_{\pm}) \Omega_n(z_{\pm}) = \delta_{mn} \quad (3)$$

where $C_+(C_-)$ denotes any contour around $P_+(P_-)$ but not including $P_-(P_+)$. In these bases, one has

$$c_{m,n}^s = \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} (\partial_{w_{\pm}} e_m(w_{\pm}) e_n(w_{\pm}) - e_m(w_{\pm}) \partial_{w_{\pm}} e_n(w_{\pm})) \Omega_{m+n-s}(w_{\pm}) \quad (4a)$$

and

$$\chi_{mn} = \frac{1}{12} \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} e_m'''(w_{\pm}) e_n(w_{\pm}). \quad (4b)$$

Similarly, the generalized Heisenberg algebra appropriate for Σ_g is defined by the relations [13]

$$[\alpha_m, \alpha_n] = \gamma_{mn}. \quad (5)$$

The structure constants are evaluated through

$$\gamma_{mn} = \frac{1}{2\pi i} \oint_{C_{\tau}} dA_m A_n \quad (6)$$

where $\{A_n\}$ is a basis of meromorphic functions on Σ_g .

Now, one of the most remarkable feature of the KN formalism is its simplicity in quantizing theories over Riemann surfaces. In fact it extends the standard operator formalism in a most natural way, removing the need for special considerations arising from the non-trivial topology. Indeed, the quantization of fields over Σ_g is carried out in the traditional way in which the coefficients of the fields in a series expansion are regarded as operators on some Fock space. Here ‘globalization’ of the theory is obtained by replacing the Laurent basis, in which fields are usually expanded, to these KN bases. The latter intrinsically carries all the topological information of the underlying surface. In the field-theoretic context, the full content of the theory is usually embodied in the operator product expansions (OPEs) of the relevant field operators. For instance, in the case of a conformal field theory, the OPE between two energy-momentum tensors bears all the information about the underlying algebra, which in the case of a sphere, is the Virasoro algebra. Although the OPEs between field operators are generically genus dependent, the singular terms are not [15]. What is interesting here is that the essential information on the algebra is contained only in the singular portion of the OPE. Using this fact, Mezincescu et al [16] showed that the KN algebra can be obtained from the OPE of $T(z)T(w)$ with the usual Laurent basis replaced by the KN basis.[†] In the following we employ similar techniques to obtain q -analogues of the generalized Heisenberg algebra and the KN algebra.

To this end, we consider a q -deformed version of a $U(1)$ current algebra and its associated conformal algebra defined on the plane. By using a q -analogue of the Sugawara construction,

$$T^{\alpha}(z) = \frac{1}{4} : J(zq^{\alpha/2})J(zq^{-\alpha/2}) : + \frac{1}{4} : J(zq^{-\alpha/2})J(zq^{\alpha/2}) : \quad (7)$$

[†] Although they consider the torus ($g = 1$) exclusively, their analysis can be extended to a surface of any genus (see ref.[17]).

it was shown in ref.[10] that the q -OPE's[‡]

$$J(z)J(w) \sim \frac{\kappa}{(z-w)_q^2} \quad (8)$$

$$\begin{aligned} T^\alpha(z)T^\beta(w) \sim & \frac{\kappa}{2(q-q^{-1})w} \left\{ \frac{T^{\alpha+\beta+1}(wq^{(\alpha+1)/2})}{(zq^{-(\alpha-\beta)/2} - wq^{\beta+1})} + \frac{T^{-\alpha+\beta-1}(wq^{(\alpha+1)/2})}{(zq^{-(\alpha+\beta)/2} - wq^{-\beta+1})} \right. \\ & - \frac{T^{-\alpha-\beta+1}(wq^{(\alpha-1)/2})}{(zq^{-(\alpha-\beta)/2} - wq^{\beta-1})} - \left. \frac{T^{\alpha-\beta-1}(wq^{(\alpha-1)/2})}{(zq^{-(\alpha+\beta)/2} - wq^{-\beta-1})} \right\} \\ & + \frac{\kappa^2}{4(q-q^{-1})w^3} \left\{ \frac{1}{(q^{\alpha+\beta/2+1} - q^{-\beta/2})_q^2} \frac{1}{(zq^{-(\alpha-\beta)/2} - wq^{\beta+1})} \right. \\ & + \frac{1}{(q^{\alpha-\beta/2+1} - q^{\beta/2})_q^2} \frac{1}{(zq^{-(\alpha+\beta)/2} - wq^{-\beta+1})} \\ & - \frac{1}{(q^{\alpha+\beta/2-1} - q^{-\beta/2})_q^2} \frac{1}{(zq^{-(\alpha-\beta)/2} - wq^{\beta-1})} \\ & - \left. \frac{1}{(q^{\alpha-\beta/2-1} - q^{\beta/2})_q^2} \frac{1}{(zq^{-(\alpha+\beta)/2} - wq^{-\beta-1})} \right\} \\ & + \quad q \leftrightarrow q^{-1} \end{aligned} \quad (9)$$

where $\kappa = (q - q^{-1})/\ln q^2$ provide a realization of the q -deformed Heisenberg algebra and the corresponding q -deformed Virasoro algebra of Chaichian and Prešnajder [4].

Now for the higher genus version, we first write the $U(1)$ current generators $\{\alpha_n\}$ on Σ_g canonically as

$$\alpha_n = \frac{1}{2\pi i} \oint_{C_\tau} J(Q) A_n(Q) \quad Q \in \Sigma_g \quad (10)$$

where the contour C_τ is an ‘equal-time’ contour [13]. In the neighbourhood of the distinguished points P_\pm with $J(Q) = J(z_\pm)dz_\pm$, this takes the form of

$$\alpha_n = \oint_{C_\pm} \frac{dz_\pm}{2\pi i} J(z_\pm) A_n(z_\pm) \quad (11)$$

where the contour C_τ is now replaced by C_\pm . The commutator between the generators can then be evaluated using the standard notion of radial ordering. To make this more

[‡] In the expressions, the q -product $(z-w)_q^2$ denotes $(z-wq^{-1})(z-wq)$.

explicit, we first consider the undeformed case, we have for $J(z)J(w) \sim (z-w)^{-2}$

$$\begin{aligned}
[\alpha_m, \alpha_n] &= \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \oint_{C_w} \frac{dz_{\pm}}{2\pi i} J(z_{\pm})J(w_{\pm})A_m(z_{\pm})A_n(w_{\pm}) \\
&= \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \oint_{C_w} \frac{dz_{\pm}}{2\pi i} \frac{A_m(z_{\pm})A_n(w_{\pm})}{(z_{\pm}-w_{\pm})^2} \\
&= \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} A'_m(w_{\pm})A_n(w_{\pm}) = \gamma_{mn}.
\end{aligned} \tag{12}$$

where C_w is a contour taken around the singular points of the OPE of $J(z)$ and $J(w)$. For the q -deformed case,[†] we simply replace the OPE by its q -analogue (8):

$$\begin{aligned}
[\alpha_m, \alpha_n] &= \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \oint_{C_w} \frac{dz_{\pm}}{2\pi i} \frac{A_m(z_{\pm})A_n(w_{\pm})}{(z_{\pm}-w_{\pm})_q^2} \\
&= \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \left(\partial_{w_{\pm}}^q A_m(w_{\pm}) \right) A_n(w_{\pm}) = \gamma_{mn}^q
\end{aligned} \tag{13}$$

where

$$\partial_{w_{\pm}}^q A_m(w_{\pm}) \equiv \frac{A_m(w_{\pm}q) - A_m(w_{\pm}q^{-1})}{w_{\pm}(q - q^{-1})}. \tag{14}$$

It is interesting to note that in the $q \rightarrow 1$ limit, one has $\partial_{w_{\pm}}^q A_m(w_{\pm}) \rightarrow \partial_{w_{\pm}} A_m(w_{\pm})$ and consequently $\gamma_{mn}^q \rightarrow \gamma_{mn}$.

Next let us consider the q -analogue of the Virasoro algebra. By writing

$$T^{\alpha}(z_{\pm}) = \sum_n \Omega_n(z_{\pm}) \mathcal{L}_n^{\alpha} \tag{15a}$$

$$\mathcal{L}_m^{\alpha} = \pm \oint_{C_{\pm}} \frac{dz_{\pm}}{2\pi i} e_n(z_{\pm}) T^{\alpha}(z_{\pm}) \tag{15b}$$

the commutator between the generators $\{\mathcal{L}_m^{\alpha}\}$ can be obtained from the q -OPE (9). To evaluate this, it is instructive to first examine the operator part of the q -OPE (*i.e.* terms involving the energy-momentum tensor). To this end we consider the generic term

$$\frac{\kappa}{2(q - q^{-1})} \frac{T^a(wq^b)}{w(zq^c - wq^d)} + q \leftrightarrow q^{-1} \tag{16}$$

where the indices a, b, c, d stand for the different algebraic terms involving α and β in the q -OPE of (9). The relevant quantity to be computed here is

$$\frac{\kappa}{2(q - q^{-1})} \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \oint_{C_w} \frac{dz_{\pm}}{2\pi i} \frac{e_m(z_{\pm})e_n(w_{\pm})T^a(w_{\pm}q^b)}{w(z_{\pm}q^c - w_{\pm}q^d)} + q \leftrightarrow q^{-1}. \tag{17}$$

[†] q is assumed to be real here.

By using (15a) and carrying out the z -integration this reduces to

$$\frac{\kappa}{2(q - q^{-1})} \sum_k \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} q^{-c} \frac{e_m(w_{\pm} q^{d-c}) e_n(w_{\pm}) \Omega_k(w_{\pm} q^b)}{w_{\pm}} \mathcal{L}_k^a + q \leftrightarrow q^{-1}. \quad (18)$$

which upon including the $q \leftrightarrow q^{-1}$ term leads to[†]

$$\begin{aligned} \frac{\kappa}{2} \sum_k \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \{ & q^{-c} [d - b - c]_q \partial_{w_{\pm}}^{q^{d-b-c}} e_m(w_{\pm}) e_n(w_{\pm} q^{-b}) \\ & - q^{-c} [b]_q e_m(w_{\pm} q^{b+c-d}) \partial_{w_{\pm}}^{q^{-b}} e_n(w_{\pm}) \\ & - [c]_q \frac{e_m(w_{\pm} q^{b+c-d}) e_n(w_{\pm} q^b)}{w_{\pm}} \} \Omega_k(w_{\pm}) \mathcal{L}_k^a. \end{aligned} \quad (19)$$

The sum over k in (19) is not infinite but runs over $(2g_0 + 1)$ terms (*i.e.* from $m + n - g_0$ to $m + n + g_0$). This can be easily shown by evaluating the w -integral around the points P_+ and P_- using the explicit form of the basis elements. Then by denoting

$$\begin{aligned} \mathcal{D}_{m,n}^s(b, c, d) \equiv \frac{\kappa}{2} \oint_{C_{\pm}} \frac{dw_{\pm}}{2\pi i} \{ & q^{-c} [d - b - c]_q \partial_{w_{\pm}}^{q^{d-b-c}} e_m(w_{\pm}) e_n(w_{\pm} q^{-b}) \\ & - q^{-c} [b]_q e_m(w_{\pm} q^{b+c-d}) \partial_{w_{\pm}}^{q^{-b}} e_n(w_{\pm}) \\ & - [c]_q \frac{e_m(w_{\pm} q^{b+c-d}) e_n(w_{\pm} q^b)}{w_{\pm}} \} \Omega_{m+n-s}(w_{\pm}). \end{aligned} \quad (20)$$

the operator part of the commutation relations can be written as

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] = \sum_{s=-g_0}^{g_0} \{ & \mathcal{D}_{m,n}^s(\frac{\alpha+1}{2}, \frac{\beta-\alpha}{2}, \beta+1) \mathcal{L}_{m+n-s}^{\alpha+\beta+1} \\ & + \mathcal{D}_{m,n}^s(\frac{\alpha+1}{2}, \frac{-\beta-\alpha}{2}, -\beta+1) \mathcal{L}_{m+n-s}^{-\alpha+\beta+1} \\ & - \mathcal{D}_{m,n}^s(\frac{\alpha-1}{2}, \frac{\beta-\alpha}{2}, \beta-1) \mathcal{L}_{m+n-s}^{-\alpha-\beta+1} \\ & - \mathcal{D}_{m,n}^s(\frac{\alpha-1}{2}, \frac{-\beta-\alpha}{2}, -\beta-1) \mathcal{L}_{m+n-s}^{\alpha-\beta-1} \} \\ & + \text{central term}. \end{aligned} \quad (21)$$

Using some of the techniques above, it is not difficult to show that the central term is given by

$$\chi_{m,n}^q = -\frac{1}{16(\ln q)^2} \sum_{k=0}^{2g_0-m-n} e_{m,k}^+ e_{n,2g_0-m-n-k}^+ \left\{ C_{m+1}^{\alpha,\beta}(g_0; k) + C_{m+1}^{\alpha,-\beta}(g_0; k) \right\} \quad (22)$$

[†] The q -bracket $[x]_q$ denotes $\frac{q^x - q^{-x}}{q - q^{-1}}$.

where

$$C_m^{\alpha,\beta}(g_0; k) = \frac{[\frac{\alpha+\beta+2}{2}(m-g_0+k)]_q}{[\frac{\alpha+\beta+2}{2}]_q} + \frac{[\frac{\alpha+\beta-2}{2}(m-g_0+k)]_q}{[\frac{\alpha+\beta-2}{2}]_q} - (q^{(m-1-g_0+k)} + q^{-(m-1-g_0+k)}) \frac{[\frac{\alpha+\beta}{2}(m-g_0+k)]_q}{[\frac{\alpha+\beta}{2}]_q}. \quad (23)$$

It should be noted that the coefficients $e_{m,k}^+$ can be evaluated for a given basis of vector fields through

$$e_{m,k}^+ = \oint_{C_+} \frac{dz}{2\pi i} e_m(z_+) z^{-m-k+g_0-1}. \quad (24)$$

For instance in the $g = 1$ case this can be computed explicitly for the vector fields $e_n = e_n(z)\partial/\partial z$ with $\{e_n(z)\}$ given in terms of the well studied elliptic functions [18]:

$$e_n(z) = \frac{\sigma^{n-1/2}(z-z_0)\sigma(z+2nz_0)}{\sigma^{n+1/2}(z+z_0)} \frac{\sigma^{n+1/2}(2z_0)}{\sigma((2n+1)z_0)} \quad n \neq -1/2 \quad (25a)$$

$$e_{-1/2}(z) = \frac{\sigma^2(z)}{\sigma(z+z_0)\sigma(z-z_0)} \frac{\sigma(2z_0)}{\sigma^2(z_0)}. \quad (25b)$$

Here $\sigma(z)$ is the Wierstrass sigma-function. It should also be noted that, by computing $\chi_{m,n}^q$ in the vicinity of P_- , the central term obeys the cocycle condition

$$\chi_{m,n}^q = 0 \quad \text{for} \quad |m-n| > 2g_0. \quad (26)$$

Finally in the $q \rightarrow 1$ limit the above algebra reduces to the usual KN algebra. Indeed, from (7) and (15b) one can ascertain that $\mathcal{L}_m^\alpha \xrightarrow{q \rightarrow 1} \mathcal{L}_m$ and the operator coefficients in (21) reduce to (4a) while the central term becomes

$$\chi_{m,n}^q \xrightarrow{q \rightarrow 1} \chi_{m,n} = \frac{1}{12} \sum_{k=0}^{2g_0-m-n} e_{m,k}^+ e_{n,2g_0-m-n-k}^+ (m-g_0+k-1)(m-g_0+k)(m-g_0+k+1) \quad (27)$$

which is equivalent to (4b) if the latter is evaluated using the basis vector fields (2a).

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